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PARTIAL INTERACTION BETWEEN ELASTICALLY CONNECTED ELEMENTS OF A COMPOSITE BEAM

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Abstract—This paper deals mainly with interaction of the elements of composite beams of steel and concrete. An interaction theory which takes account of slip, uplift and friction effects is formulated without, as in existing theories, assuming equal curvatures of the interacting elements. The two resulting simultaneous differential equations connecting the uplift tension arising from differential deflexions of the two elements with the axial force within each of the elements are solved by a finite-difference approach.

The results of computations for a typical composite section, for the case of zero interface coefficient of friction but based on experimentally determined shear connection and foundation moduli are presented.

NOTATION

a_c	cross-sectional area of a connector
d	half depth of lower element (symmetrical steel joint)
e_a	common interface strain in the top element
e_b	common interface strain in the lower element
α_a	$= (E_a A_a)$ modulus of elasticity × area; top element
α_a	$= (E_b A_b)$ modulus of elasticity × area; lower element
β_a	$= (E_a I_a)$ flexural rigidity of top element
β_b	$= (E_b I_b)$ flexural rigidity of lower element
F	axial force in the top or bottom element
h	distance between centroidal axes of top and bottom elements
I	second moment of area
k_s	shear connexion modulus (i.e. force per unit slip per unit length)
k_t	uplift tension modulus or foundation modulus (i.e. force per unit uplift per unit length)
L	span of simply supported beam
M_a	moment in top element
M_b	moment in bottom element
M	external applied moment
Q_a	shear in the top element
Q_b	shear in the lower element
T	uplift force per unit length
t	half depth of top element
V	external shear
w	intensity of applied uniformly distributed loading
x	longitudinal axis
y	vertical deflexion
Z_a	distance from neutral axis of the top element
Z_b	distance from neutral axis of the bottom element
γ	slip at the interface of the elements
τ	shear per unit length
σ	direct stress
μ	coefficient of friction

INTRODUCTION

WHEN two elements capable of resisting bending moments are elastically connected together at an interface, interaction, partial or complete, between the two elements takes place. Where the elastic connexion is flexible differential direct strains at the common interface exist resulting in slip, and differential deflexions may also result giving rise to uplift between the two elements.

Earlier works on this subject [1–3] have attempted to deal with interaction in composite beams of steel and concrete and in a ship's hull and its superstructure. In this paper, however, only interaction of composite beams of steel and concrete is considered. The formulations of existing works take account of either differential strains only, or of differential deflexions only, but not both together. Observations on composite beams [4] of steel and concrete indicate that both slip and uplift occur simultaneously where the elastic connexion is flexible.

A single theory of interaction taking both slip and uplift effects into account is presented assuming bending theory but ignoring shear lag effects. Differential equations governing the relation between uplift forces and the axial forces are formulated for the region of positive uplift as well as the region of negative uplift for which frictional effects are incorporated. This friction arises as a result of the pressure (i.e. negative uplift) at the support ends of the beam and the sliding at the interface of the two elements.

FORMULATION OF THE THEORY

Each element of a composite member is assumed to behave separately in accordance with simple bending theory, so that the longitudinal stress distribution over the depth of the entire composite section is not necessarily colinear. In addition it is assumed that the rate of change of the axial force is directly proportional to slip, and uplift force is directly proportional to differential deflexion. This last assumption implies the existence of two moduli, one which depends on the ability of the connectors to resist slip and the other which depends on the resistance of the connectors to uplift in regions where separation occurs, or the compressibility of the concrete bearing on the steel joist where uplift is negative. A difficulty arises in estimating foundation modulus in respect of negative uplift for which no experimental values exist and also in knowing beforehand over what portion of a beam this negative uplift will exist.

Foundation modulus and shear modulus

Shear connexion modulus has been shown in push-out tests to depend on the properties and dimensions of shear connectors as well as the properties of the surrounding concrete. Usually in the analysis of composite beams a continuous shear modulus is obtained by dividing an experimentally determined modulus for the type of connector to be used by the discreet longitudinal spacing of the connectors and multiplied by the number of connectors at the cross-section considered. Foundation modulus in respect of positive uplift depends largely on the elastic properties and the dimensions and spacing of the connectors and is usually determined experimentally from pull out tests of shear connectors embedded in concrete [5].

Frictional effects

Frictional resistance in the region of negative uplift will contribute to shear resistance at the common interface thereby reducing the load on the shear connectors in that region. This tendency will be reflected in reduced magnitudes of slip in these regions. However the portion of the beam over which friction is operative cannot be uniquely predetermined so that an iterative procedure assuming zero coefficient of friction initially would be necessary. The solution for zero coefficient of friction will indicate approximately the regions of negative uplift and the calculation then repeated introducing the frictional coefficient in the appropriate regions in the finite difference relations discussed under Method of Solution.

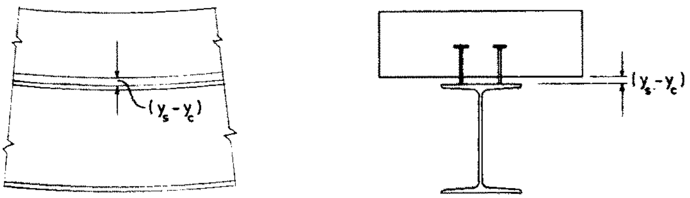

FIG. 1

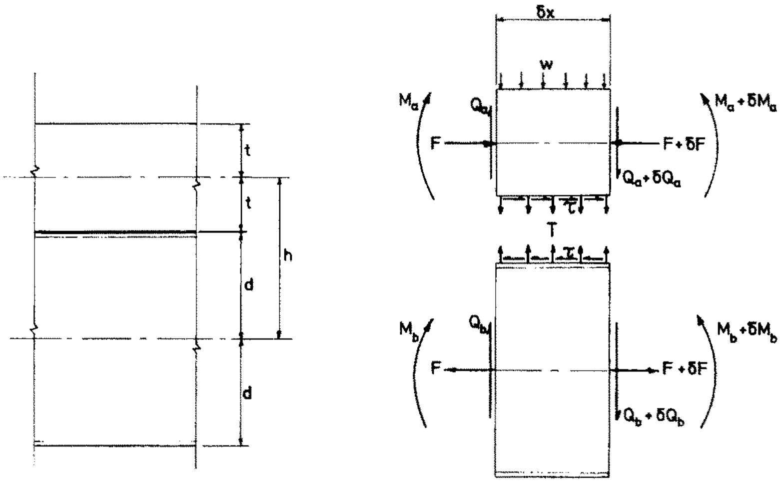

FIG. 2

Consider the equilibrium of an element of length δx of the composite section shown in Fig. 2. The equations of equilibrium are as given below

Upper concrete element		Lower steel element	
$\delta Q_a = -(w+T)\delta x$		$\delta Q_b = T\delta x$	
$\delta F = \tau\delta x$	(1a)	$\delta F = \tau\delta x$	(1b)
$\delta M_a = Q_a\delta x - t\delta F$		$\delta M_b = Q_b\delta x - d\,.\,\delta F.$	

Furthermore the external moment at any cross-section of the beam will be resisted by the sum of the moments in the upper and lower elements plus the couple arising from the axial forces in the two elements. Hence

$$M = M_a + M_b + F \, . \, h \tag{2}$$

From the last of equations (1a) and (1b),

$$\frac{d^2 M_a}{dx^2} = \frac{dQ_a}{dx} - t \, . \, \frac{d^2 F}{dx^2}, \qquad \frac{d^2 M_b}{dx^2} = \frac{dQ_b}{dx} - d \, . \, \frac{d^2 F}{dx^2}.$$

Writing the bending moments in terms of deflections and substituting for Q these equations become

$$-\beta_a \frac{d^4 y_a}{dx^4} = -(w+T) - t \, . \, \frac{d^2 F}{dx^2} \tag{3a}$$

$$-\beta_b \frac{d^4 y_b}{dx^4} = T - d \, . \, \frac{d^2 F}{dx^2}. \tag{3b}$$

The tension T arises from the deformation of the elastic connexion due to differential displacement between the two elements at their interface, so that

$$T = k_t(y_b - y_a). \tag{4}$$

Differentiating equation (4) four times with respect to x and substituting for $d^4 y_a/dx^4$ and $d^4 y_b/dx^4$ from equations (3a) and (3b) yields

$$\frac{d^4 T}{dx^4} + k_t \left\{ \frac{1}{\beta_a} + \frac{1}{\beta_b} \right\} T - k_t \left\{ \frac{d}{\beta_b} - \frac{t}{\beta_a} \right\} . \frac{d^2 F}{dx^2} + \frac{k_t w}{\beta_a} = 0. \tag{5}$$

Again differentiating equation (4) twice and substituting for curvatures in terms of moments and flexural rigidities and rearranging gives

$$-\frac{M_b}{\beta_b} + \frac{M_a}{\beta_a} = \frac{1}{k_t} \, . \, \frac{d^2 T}{dx^2}. \tag{6}$$

From equations (2) and (6) the moments in the upper and lower elements are

$$\frac{M_a}{\beta_a} = \left\{ \frac{M - Fh}{\beta_b} + \frac{1}{k_t} \, . \, \frac{d^2 T}{dx^2} \right\} . \frac{\beta_b}{\beta_a + \beta_b} \tag{7a}$$

$$\frac{M_b}{\beta_b} = \left\{ \frac{M - Fh}{\beta_a} - \frac{1}{k_t} \, . \, \frac{d^2 T}{dx^2} \right\} . \frac{\beta_a}{\beta_a + \beta_b}. \tag{7b}$$

The rate of change of slip at the common interface at any point is equal to the differential strain at that point. Hence

$$\frac{d\gamma}{dx} = e_b - e_a = \frac{F}{\alpha_b} - \frac{M_b d}{\beta_b} + \frac{F}{\alpha_a} - \frac{M_a t}{\beta_a}$$

$$= \left\{ \frac{1}{\alpha_a} + \frac{1}{\alpha_b} + \frac{h^2}{\beta_a + \beta_b} \right\} . F - \frac{h}{\beta_a + \beta_b} . M + \frac{[d\beta_a - t\beta_b]}{k_t(\beta_a + \beta_b)} \, . \, \frac{d^2 T}{dx^2}. \tag{8}$$

The axial force F transferred from the lower element to the upper element by means of the elastic shear connexion is given by $F = \int k_s\gamma\,\mathrm{d}x$, for regions of positive uplift, or $F = \int k_s\gamma\,\mathrm{d}x + \int \mu C\,\mathrm{d}x$, for regions of negative uplift, where C is the negative uplift force/unit length of beam in the negative uplift regions and equals T. Hence

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = k_s\frac{\mathrm{d}\gamma}{\mathrm{d}x} \tag{9a}$$

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = k_s\frac{\mathrm{d}\gamma}{\mathrm{d}x} + \mu\frac{\mathrm{d}T}{\mathrm{d}x} \tag{9b}$$

according as whether the uplift is positive or negative.

Substituting for $\mathrm{d}\gamma/\mathrm{d}x$ from equation (8) and rearranging gives

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} - k_s\left\{\frac{1}{\alpha_a}+\frac{1}{\alpha_a}+\frac{h^2}{\beta_a+\beta_b}\right\}.F - \frac{k_s}{k_t}\left\{\frac{d\beta_a - t\beta_b}{\beta_a+\beta_b}\right\}.\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - \boxed{\mu\frac{\mathrm{d}T}{\mathrm{d}x}} = -\frac{k_sh}{\beta_a+\beta_b}.M. \tag{10}$$

The expression within the square being incorporated only in regions of negative uplift tension.

Equations (5) and (10) give the complete solutions for axial force and uplift force from which slip, differential deflexions and stresses can be determined.

If k_t is infinite, uplift tensions will not be determinable from equation (5) since, for T to be finite, $y_b - y_a$ must be zero and equation (10) becomes the well known Newmark equation.

Deflexions can be found by using equation (3a) or (3b) together with equation (4).

The stresses are given by

$$\frac{\sigma_a}{E_a} = \frac{F}{\alpha_a} \pm \left\{\frac{M-Fh}{\beta_b} + \frac{1}{k_t}.\frac{\mathrm{d}^2T}{\mathrm{d}x^2}\right\}.\frac{\beta_b}{\beta_a+\beta_b}Z_a \tag{11}$$

$$\frac{\sigma_b}{E_b} = \frac{F}{\alpha_b} \pm \left\{\frac{M-Fh}{\beta_a} - \frac{1}{k_t}.\frac{\mathrm{d}^2T}{\mathrm{d}x^2}\right\}.\frac{\beta_a}{\beta_a+\beta_b}Z_b. \tag{12}$$

METHOD OF SOLUTION

The two equations connecting uplift tension and axial force are solved by a method suggested by Fox [6] for solving two-point boundary value problems involving differential equations of orders higher than two. In order to achieve good accuracy in the solution of the differential equations by finite differences, the equations are rearranged such that no derivative higher than the second order occurs. Thus equation (5) is modified by assuming a function.

$$U = \mathrm{d}^2T/\mathrm{d}x^2 \tag{13}$$

so that this equation becomes

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + k_t\left\{\frac{1}{\beta_a}+\frac{1}{\beta_b}\right\}T. - k_t\left\{\frac{d}{\beta_b}-\frac{t}{\beta_a}\right\}.\frac{\mathrm{d}^2F}{\mathrm{d}x^2} + k_t.\frac{w}{\beta_a} = 0 \tag{14}$$

whilst equation (10) takes the form

$$\frac{d^2F}{dx^2}-k_s\left\{\frac{1}{\alpha_a}+\frac{1}{\alpha_b}+\frac{h}{\beta_a+\beta_b}\right\}.F-\frac{k_s}{k_t}\left\{\frac{d\beta_a-t\beta_b}{\beta_a+\beta_b}\right\}U-\boxed{\mu.\frac{dT}{dx}}=-\frac{k_sh}{\beta_a+\beta_b}.M. \quad (15)$$

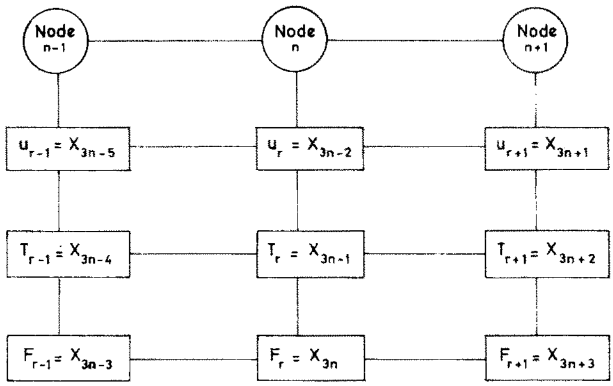

FIG. 3

With this rearrangement there will be three unknowns U, T and F for each node point of a composite beam divided up in regular intervals.

If the three unknowns at each node are all designated by x's say, then for a central node point number, n, and its two adjacent node points, $n-1$ and $n+1$, the numbering pattern of Fig. 3 would emerge for insertion into finite difference relations for the governing differential equations (13), (14) and (15).

The pattern of the resulting matrix of the finite difference relations embodying the coefficients of the unknown's is a square matrix arranged in a diagonal band of eight coefficients (except for the first three and last three rows), the other elements being zero elements. The right hand side of the matrix relation involves the values of the known external bending moment and the boundary conditions.

Boundary conditions

In the case of a simply supported system where the length of the upper elements is the same as that of the lower the following conditions apply:

(i) The moments in the upper and lower elements are both zero at the supports; i.e. at $x = 0$ and $x = L$, $U = 0$

(ii) By differentiating equation (4) thrice, the following relation results

$$\frac{1}{k_t}.\frac{d^3T}{dx^3}=\frac{d^3y_b}{dx^3}-\frac{d^3y_a}{dx^3}$$

$$=\frac{1}{\beta_b}.\frac{dM_b}{dx}-\frac{1}{\beta_a}.\frac{dM_a}{dx}$$

$$=\frac{Q_b}{\beta_b}-\frac{Q_a}{\beta_a}.$$

Since the slab is connected to the steel beam by means of a continuous foundation modulus, then at the supports the steel beam would resist all of the external shear force at that point only. Hence $Q_b = V$, so that at $x = 0$ and $x = L$, $dU/dx = k_t V/\beta_b$. It must be borne in mind however that in a simply supported system V would be of opposite signs for the two supports.

Computation

The dimensions of the section adopted for the computations are as follows:

$$t = 3\text{ in.},\quad d = 6\text{ in.},\quad A_c = 288\text{ in}^2,\quad A_s = 13\text{ in}^2,$$
$$I_c = 864\text{ in}^4,\quad I_s = 317\text{ in}^4,\quad L = 216\text{ in.},\quad m = 6{\cdot}5,$$
$$E_s = 13{,}000\text{ ton/in}^2.$$

These are the dimensions of some of the test specimens of Balakrishnan for which he gave the value of foundation modulus from pull-out tests on $\frac{3}{4}$-in. dia. studs as 446 ton/in and the corresponding value of shear modulus as 1000 ton/in for studs spaced at 8·5 in. centres in pairs. For the purpose of computation it is assumed that the two moduli bear a constant relation for different stud spacing which would affect the value of c and k_t. From Balakrishnan's test results $c = 0{\cdot}0833$ and $k_t/E_s = 40{\cdot}4 \times 10^{-4}$. c is a nondimensional

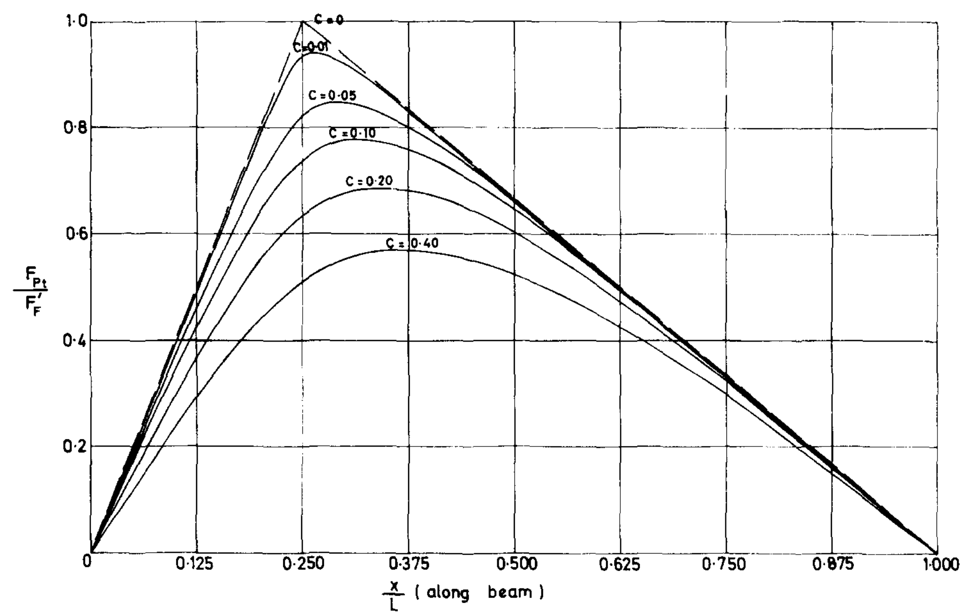

FIG. 4. Influence of c on interaction (point load at quarter span).

parameter which depends on the properties of the composite section as well as the shear modulus of the connexion and is given by

$$c = \frac{1}{k_s} \cdot \frac{\pi^2}{l^2} \cdot \frac{1}{\dfrac{1}{\alpha_a} + \dfrac{1}{\alpha_b} + \dfrac{h^2}{\beta_a \quad \beta_b}}.$$

Figure 4 gives the degree of interaction curves for various values of c. The curves are the same as those obtained from Newmarks theory which ignored uplift effects.

Figure 5 gives the slip distribution and Fig. 6 the uplift distribution along the length of a beam for a point load at midspan, for $c = 0{\cdot}1$ and $k_t/E_s = 33{\cdot}6 \times 10^{-4}$ which represent a slightly less degree of interaction than the values obtained from Balakrishnan's test results.

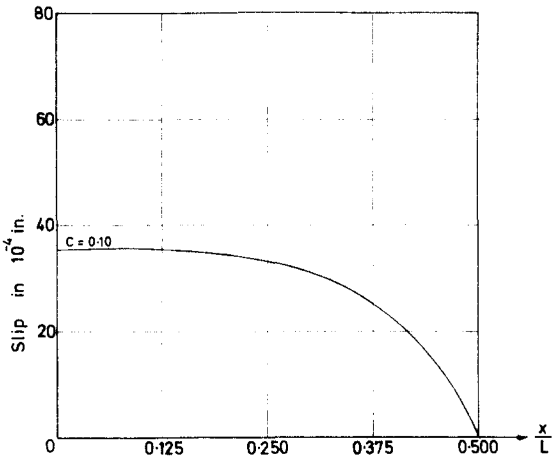

FIG. 5. Slip distribution for point load at midspan.

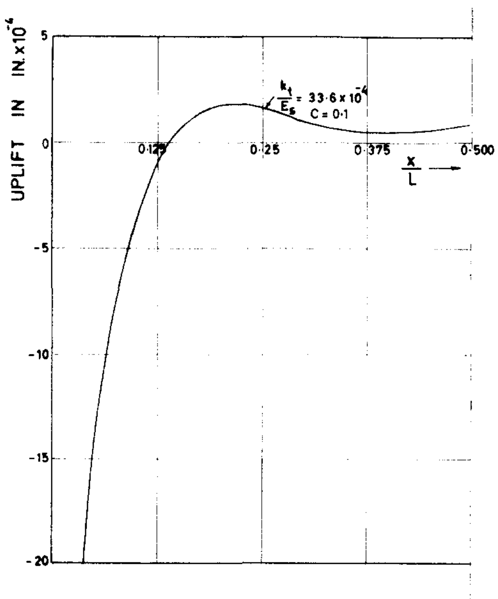

FIG. 6. Uplift distributions for point load at midspan.

Figures 7 and 8 give similar distributions for the case of a point load at $\frac{1}{4}$ span point.

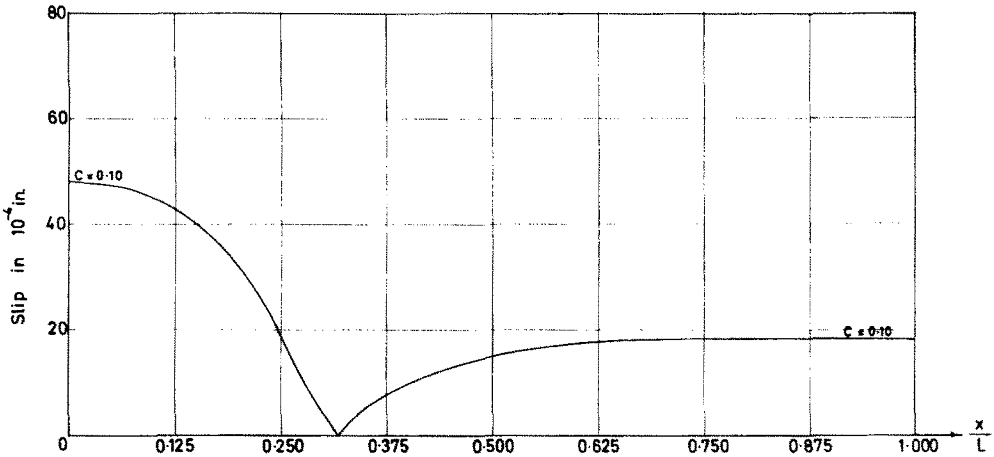

FIG. 7. Slip distribution for point load at quarter span.

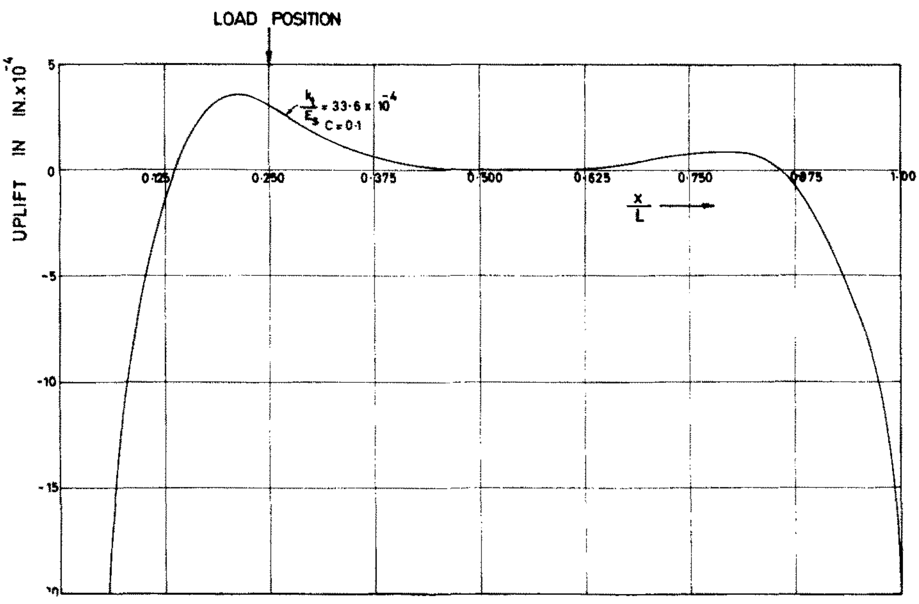

FIG. 8. Uplift distributions for point load at quarter span.

Uplift distributions obtained from the analysis are similar to the ones obtained by Gogoi [5] who took account only of differential deflexion.

In the case of composite beams, there arises the problem of estimating the true foundation modulus to use in computations for regions of negative uplift. The current analysis assumes a constant value of foundation modulus throughout the length of a beam which has resulted in very high negative uplifts near the supports. Although negative uplifts had been recorded near the supports of simply supported beams in experiments, magnitudes of experimental values are much less than the theory predicts.

Figure 9 shows the ratio of steel bottom flange stress at partial interaction with steel bottom flange stress at full interaction for three loading cases. The adverse effect on steel bottom flange stress of partial interaction is least under uniformly distributed loading and highest under quarter span point loading. Also for a high degree of interaction to be achieved in respect of bending stress c must be of the order 0·01. This represents a high shear modulus which may not be easily realised in practice.

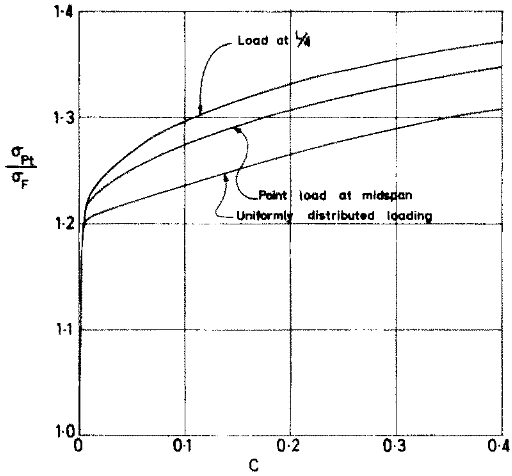

FIG. 9. Variation of maximum steel bottom flange stress as a ratio of maximum steel bottom flange stress for full interaction with c.

CONCLUSION

A theory is given which permits the analysis of composite beam elements without making the usual simplifying assumptions of previous theories. If shear lag effects are neglected it is now possible to undertake an analytical study of partial interaction in composite beams that would take into account effects of differential deflexion as well as differential strains at the common interface and interface friction in one theory. The method of computation would permit the use of variable shear foundation modulus along the length of a beam.

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Абстракт—Работа занимается, главным образом, элементами балок состоящих из стали и бетона. Приводитсн теория взаимодейстия, которая принимает во внимание зффекты скольжения, взброса и сцепления, но не учитывает равных кривизн взаимодействия элементов, как это было в существующих, в настоящее время, теориях. Система двух результирующих дифференциальных уравнеинй, касающаяся растяжения при взбросе, являющегося резулвтатом дифференциальмых прогибов лвух элементов, под влиянием осевой силы внутри каждого элемента, определяется с помощью метода конечных разностей.

Представляются результаты расчетов для типичной составной секции. Они касаются случая нулевого коэффициента сцепления и определяются на найденном экспериментально влючению сдвига и модулях основания.